# Integration by Parts

## Worksheet Solutions

1. Evaluate the following indefinite integrals by parts.

a)
$$\int x\cos(x)\,dx$$

b)
$$\int -4xe^{4x}\,dx$$

c)
$$\int \ln(x)\,dx$$

d)
$$\int (x^4-2)\ln(x)\,dx$$

e)
$$\int e^x\cos(2x)\,dx$$

f)
$$\int x^3\sin(x)\,dx$$

a) $\int x \cos x \, dx$

let $u = x$ , let $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1$ , $v = \sin x$

$$\int u \frac{dv}{dx}\,dx = uv - \int \frac{du}{dx} v \,dx$$

$$\Rightarrow \int x \cos x \,dx = x\sin x - \int 1(\sin x)\,dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C.$$

b) $\int -4x \, e^{4x} \, dx$

let $u = -4x$ , let $\frac{dv}{dx} = e^{4x}$

$\frac{du}{dx} = -4$ , $v = \frac{1}{4} e^{4x}$

$$\Rightarrow \int -4x \, e^{4x} \, dx = -4x \cdot \frac{1}{4} e^{4x} - \int -4 \cdot \frac{1}{4} e^{4x} \, dx$$

$$= -x e^{4x} + \int e^{4x} \, dx$$

$$= -x e^{4x} + \frac{1}{4} e^{4x} + C$$

$$= e^{4x} \left( \frac{1}{4} - x \right) + C.$$

c) $\int \ln(x) \, dx$

let $u = \ln(x)$ , let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ , $v = x$

$\Rightarrow \int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx$

$= x \ln x - \int dx$

$= x \ln x - x + C.$

d) $\int (x^4 - 2) \ln(x) \, dx$

let $u = \ln x$ , let $\frac{dv}{dx} = x^4 - 2$

$\frac{du}{dx} = \frac{1}{x}$ , $v = \frac{1}{5}x^5 - 2x$

$\Rightarrow$

$\int (x^4-2)\ln(x)\, dx = (\frac{1}{5}x^5 - 2x)\ln x - \int \frac{1}{5}x^4 - 2 \, dx$

$= x \ln x (\frac{1}{5}x^4 - 2) - \frac{1}{25}x^5 + 2x + C$

$= x((\frac{1}{5}x^4 - 2)\ln x - \frac{1}{25}x^4 + 2) + C.$

e) $\int e^{x} \cos(2x)\, dx = I$

let $u = \cos(2x)$ , let $\frac{dv}{dx} = e^{x}$

$\frac{du}{dx} = -2\sin 2x$ , $v = e^{x}$

$\Rightarrow \int e^{x} \cos(2x)\, dx$

$= e^{x}\cos(2x) + 2\int e^{x} \sin 2x \, dx$

Use parts again

let $u = \sin 2x$ let $\frac{dv}{dx} = e^{x}$

$\frac{du}{dx} = 2\cos 2x$ , $v = e^{x}$

Hence $\int e^{x} \sin 2x \, dx = e^{x}\sin 2x - 2\int e^{x}\cos 2x \, dx$

Putting this all together:

$I = \int e^{x}\cos 2x \, dx = e^{x}\cos 2x + 2\left[e^{x}\sin 2x - 2\int e^{x}\cos 2x \, dx\right]$

$\Rightarrow$

$$I = e^x \cos 2x + 2e^x \sin 2x - 4I$$

$$\Rightarrow \quad 5I = e^x \cos 2x + 2e^x \sin 2x$$

$$\Rightarrow I = \int e^x \cos 2x \, dx = \frac{1}{5} e^x \cos 2x + \frac{2}{5} e^x \sin 2x.$$

This technique is absolutely gorgeous!
(unlike my handwriting!)

d) $\int x^3 \sin(x) \, dx$ we're going to end up applying parts 3 times!

let $u = x^3$, let $\frac{dv}{dx} = \sin x$

$u' = 3x^2$, $v = -\cos x$

$$\Rightarrow \int x^3 \sin x \, dx = -x^3 \cos x + 3\int x^2 \cos x \, dx$$

let $u = x^2$, let $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2x$, $v = \sin x$

$$\Rightarrow \int x^3 \sin x \, dx = -x^3 \cos x + 3\left[x^2 \sin x - 2\int x \sin x \, dx\right]$$

$$= -x^3 \cos x + 3x^2 \sin x - 6\int x \sin x \, dx.$$

let $u = x$ , let $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1$ , $v = -\cos x$

$$\Rightarrow \int x^3 \sin x \, dx$$

$$= -x^3 \cos x + 3x^2 \sin x - 6\left[-x \cos x + \int \cos x \, dx\right]$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6\int \cos x \, dx$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Notice any patterns? What form would $\int x^n \sin x \, dx$ take?

2. Evaluate the following definite integrals by parts.

a)

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 4x \sec^2(3x)\, dx$$

b)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2x \sin x \, dx$$

c)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos(x)\, dx$$

d)

$$\int_{-4}^{2} x^2 e^{\frac{x}{2}}\, dx$$

e)

$$\int_{2}^{4} \ln(x^2 - 1)\, dx$$

f)

$$\int_{\frac{1}{2}}^{1} \frac{\ln x}{x^2}\, dx$$

a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 4x \sec^2(3x)\, dx$

let $u = 4x$ , let $\frac{dv}{dx} = \sec^2(3x)$

$\frac{du}{dx} = 4$ , $v = \frac{1}{3}\tan(3x)$

$$\Rightarrow \int 4x \sec^2(3x)\, dx = \left[\frac{4}{3} x \tan 3x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \frac{4}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan(3x)\, dx$$

Consider $\int \tan 3x \, dx = -\int \frac{-\sin 3x}{\cos 3x}\, dx$

let $u = \cos 3x \Rightarrow du = -3\sin x \, dx$

$$\Rightarrow \int \tan 3x \, dx = -\int \frac{1}{3}\frac{1}{u}\, du$$

$$= -\frac{1}{3}\ln|u| + C$$

$$= -\frac{1}{3}\ln|\cos 3x| + C.$$

Hence $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 4x \sec^2 3x \, dx$

$$= \left[\frac{4}{3}x\tan 3x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} - \frac{4}{3}\left[-\frac{1}{3}\ln|\cos 3x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{4}{3}\left[x\tan 3x + \frac{1}{3}\ln|\cos 3x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{4}{3}\left(\left(\frac{\pi}{3}\tan\pi + \frac{1}{3}\ln|\cos\pi|\right) - \left(\frac{\pi}{4}\tan\frac{3\pi}{4} + \frac{1}{3}\ln\left|\cos\frac{3\pi}{4}\right|\right)\right)$$

$$= \frac{4}{3}\left(\left(\frac{\pi}{3}\cdot 0 + \frac{1}{3}\ln|-1|\right) - \left(-\frac{\pi}{3}(-1) + \frac{1}{3}\ln\left|-\frac{1}{\sqrt{2}}\right|\right)\right)$$

$$= \frac{4}{3}\left(\frac{1}{3}\ln 1 - \frac{\pi}{3} - \frac{1}{3}\ln\frac{\sqrt{2}}{2}\right)$$

$$= \frac{4}{3}\left(\frac{1}{3}\ln\sqrt{2} - \frac{\pi}{3}\right)$$

$$= \frac{4}{9}\left(\ln\sqrt{2} - \pi\right).$$

b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2x \sin x \, dx$

let $u = 2x$ , let $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2$ $v = -\cos x$

$$\Rightarrow \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2x \sin x \, dx$$

$$= \Big[-2x\cos x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, dx$$

$$= \Big[-2x\cos x + 2\sin x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left(-\frac{\pi}{2}\cos\frac{\pi}{4} + 2\sin\frac{\pi}{4}\right) - \left(\frac{\pi}{2}\cos -\frac{\pi}{4} + 2\sin -\frac{\pi}{4}\right)$$

$$=\left(-\frac{\pi}{2}\cdot\frac{\sqrt{2}}{2}+2\cdot\frac{\sqrt{2}}{2}\right)-\left(\frac{\pi}{2}\cdot\frac{\sqrt{2}}{2}+2\cdot\frac{\sqrt{2}}{2}\right)$$

$$=-\frac{\pi\sqrt{2}}{4}+\sqrt{2}-\frac{\pi\sqrt{2}}{4}+\sqrt{2}$$

$$=2\sqrt{2}-\frac{2\sqrt{2}\,\pi}{4}=2\sqrt{2}\left(1-\frac{\pi}{4}\right).$$

c) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x}\cos x\,dx$

let $u=\cos x$ , let $\frac{dv}{dx}=e^{-x}$

$\frac{du}{dx}=-\sin x$ , $v=-e^{-x}$

$\Rightarrow \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x}\cos x\,dx=$

$\left[-e^{-x}\cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x}\sin x\,dx$

let $u=\sin x$ , let $\frac{dv}{dx}=e^{-x}$

$$\frac{du}{dx} = \cos x \quad , \quad v = -e^{-x}$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos x \, dx$$

$$= \left[ -e^{-x} \cos x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \left( \left[ -e^{-x} \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos x \, dx \right)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos x \, dx = \left[ -e^{-x} \cos x + e^{-x} \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos x \, dx$$

$$= 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos x \, dx = \left[ -e^{-x} \cos x + e^{-x} \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-x} \cos x \, dx = \frac{1}{2} \left[ -e^{-x} \cos x + e^{-x} \sin x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left( \left( -e^{-\frac{\pi}{2}} \cos \frac{\pi}{2} + e^{-\frac{\pi}{2}} \sin \frac{\pi}{2} \right) - \left( -e^{\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right) + e^{\frac{\pi}{2}} \sin\left(-\frac{\pi}{2}\right) \right) \right)$$

$$= \frac{1}{2} \left( e^{-\frac{\pi}{2}} + e^{-\frac{\pi}{2}} \right) = e^{-\frac{\pi}{2}} .$$

d) $\int_{-4}^{2} x^2 e^{\frac{x}{2}}\, dx$

let $u = x^2$, let $\frac{dv}{dx} = e^{\frac{x}{2}}$

$\frac{du}{dx} = 2x$, $v = 2e^{\frac{x}{2}}$

$\Rightarrow \int_{-4}^{2} x^2 e^{\frac{x}{2}}\, dx = \left[2x^2 e^{\frac{x}{2}}\right]_{-4}^{2} - 4\int_{-4}^{2} x e^{\frac{x}{2}}\, dx$

let $u = x$, let $\frac{dv}{dx} = e^{\frac{x}{2}}$

$u' = 1$, $v = 2e^{\frac{x}{2}}$

$\Rightarrow \int_{-4}^{2} x^2 e^{\frac{x}{2}}\, dx$

$= \left[2x^2 e^{\frac{x}{2}}\right]_{-4}^{2} - 4\left(\left[2x e^{\frac{x}{2}}\right]_{-4}^{2} - 2\int_{-4}^{2} e^{\frac{x}{2}}\, dx\right)$

$= \left[2x^2 e^{\frac{x}{2}}\right]_{-4}^{2} - 4\left[2x e^{\frac{x}{2}}\right]_{-4}^{2} + 8\left[2e^{\frac{x}{2}}\right]_{-4}^{2}$

$= \left[2x^2 e^{\frac{x}{2}} - 8x e^{\frac{x}{2}} + 16 e^{\frac{x}{2}}\right]_{-4}^{2}$

$$= \left(2(4)e - 16e + 16e\right) - \left(32e^{-2} + 32e^{-2} + 16e^{-2}\right)$$

$$= 8e - 80e^{-2} = 8\left(e - \frac{10}{e^2}\right).$$

e) $\displaystyle\int_2^4 \ln(x^2)\, dx$

let $u = \ln(x^2)$ , let $\frac{dv}{dx} = 1$

$$\frac{du}{dx} = \frac{2x}{x^2} = \frac{2}{x} \quad , \quad v = x$$

$$\int_2^4 \ln(x^2)\,dx = \left[x \ln x^2\right]_2^4 - \int_2^4 2 \, dx$$

$$= \left[x \ln x^2 - 2x\right]_2^4$$

$$= (4 \ln 16 - 8) - (2 \ln 4 - 4)$$

$$= 2 \ln 256 - 8 + 2 \ln 4 + 4$$

$$= 2\left(\ln(256 \cdot 4)\right) - 4 = 2 \ln(1024) - 4.$$

f) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{\ln x}{x^2}\, dx$

let $u = \ln x$ , let $\frac{dv}{dx} = x^{-2}$

$\frac{du}{dx} = \frac{1}{x}$ , $v = -\frac{1}{x}$

$\Rightarrow \displaystyle\int_{\frac{1}{2}}^{1} \frac{\ln x}{x^2}\, dx = \left[-\frac{\ln x}{x}\right]_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1} \frac{1}{x^2}\, dx$

$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_{\frac{1}{2}}^{1}$

$= \left(-\frac{\ln 1}{1} - 1\right) - \left(-\frac{\ln\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)} - \frac{1}{\left(\frac{1}{2}\right)}\right)$

$= -1 + 2\ln\left(\frac{1}{2}\right) + 2$

$= 1 - \ln 4.$

3. (a) Differentiate $\arctan(x)$, with respect to $x$.

(b) Using the result in (a), find

$$\int \arctan(x)\, dx.$$

a) let $y = \arctan x \Rightarrow x = \tan y$

$\Rightarrow 1 = \sec^2 y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y}$

$= \frac{1}{1+x^2}$ .

Hence $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$ .

b) $\int \arctan x \; dx$

let $u = \arctan x$ , let $\frac{dv}{dx} = 1$

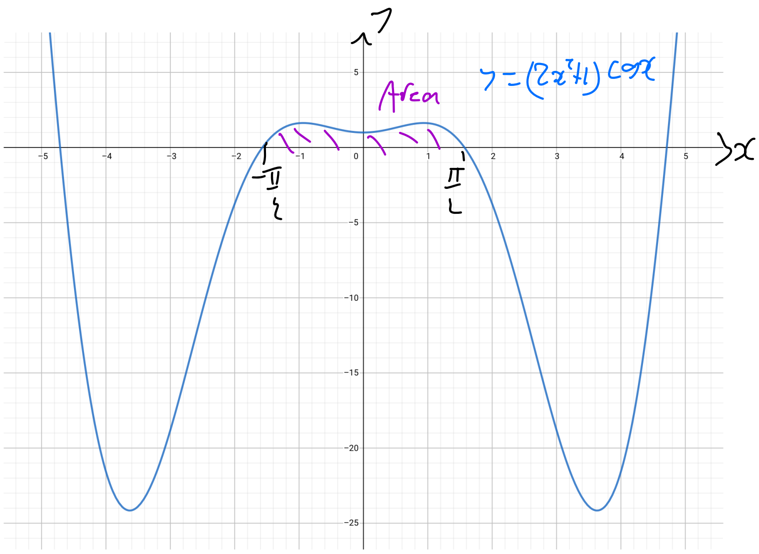
$u' = \frac{1}{1+x^2}$ , $v = x$

$\Rightarrow \int \arctan x \; dx = x \arctan x - \int \frac{x}{1+x^2} dx$

$= x \arctan x - \frac{1}{2} \ln(1+x^2) + C$ .

4. Find the area bounded by the curve $y = (2x^2+1)\cos(x)$ and the lines $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$.

Graph:




$$\text{Area} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2x^2+1)\cos x \, dx$$

let $u = 2x^2+1$, let $\frac{dv}{dx} = \cos x$

$u' = 2x$, $v = \sin x$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2x^2+1)\cos x \, dx$$

$$= \Big[(2x^2+1)\sin x\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \sin x \, dx$$

let $u = x$, let $\frac{dv}{dx} = \sin x$

$u' = 1$, $v = -\cos x$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2x^2+1)\cos x \, dx =$$

$$\left[(2x^2+1)\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - 2\left(\left[-x\cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx\right)$$

$$= \left[(2x^2+1)\sin x + 2x\cos x + \sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\left(\frac{\pi^2}{2}+1\right)\sin\left(\frac{\pi}{2}\right) + 2\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right)$$

$$- \left(\left(\frac{\pi^2}{2}+1\right)\sin\left(-\frac{\pi}{2}\right) + 2\,\frac{-\pi}{2}\cos\frac{-\pi}{2} + \sin -\frac{\pi}{2}\right)$$

$$= \left(\frac{\pi^2}{2}+1\right) + 1 + \left(\frac{\pi^2}{2}+1\right) + 1$$

$$= \pi^2 + 4 \text{ units}^2.$$

5. (a) Differentiate $\arcsin(2x)$, with respect to $x$.

(b) Using the result in (a), find:

$$\int_{-\frac{1}{2}}^{0} 4 \ \arcsin(2x)\, dx.$$

a) let $y = \arcsin 2x$

$\Rightarrow 2x = \sin y$

$\Rightarrow 2 = \cos y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2}{\cos y} = \frac{2}{\sqrt{1-\sin^2 y}}$

$= \frac{2}{\sqrt{1-(2x)^2}}$

$= \frac{2}{\sqrt{1-4x^2}}$.

Hence $\frac{d}{dx}(\arcsin 2x) = \frac{2}{\sqrt{1-4x^2}}$.

b) $\int_{-\frac{1}{2}}^{0} 4 \arcsin 2x \ dx$

let $u = \arcsin 2x$, let $\frac{dv}{dx} = 4$

$u' = \frac{2}{\sqrt{1-4x^2}}$ $\qquad v = 4x$

$$\Rightarrow \int_{-\frac{1}{2}}^{0} 4\arcsin 2x \, dx = \Big[4x \arcsin 2x\Big]_{-\frac{1}{2}}^{0} - \int_{-\frac{1}{2}}^{0} \frac{8x}{\sqrt{1-4x^2}} \, dx$$

Consider $\displaystyle\int \frac{8x}{\sqrt{1-4x^2}} \, dx$

let $u = 1-4x^2 \Rightarrow du = -8x \, dx$

$$\Rightarrow \int \frac{8x}{\sqrt{1-4x^2}} \, dx = -\int u^{-\frac{1}{2}} \, du$$

$$= -2\sqrt{u} + C = -2\sqrt{1-4x^2} + C.$$

Hence $\displaystyle\int_{-\frac{1}{2}}^{0} 4\arcsin 2x \, dx$

$$= \Big[4x \arcsin 2x + 2\sqrt{1-4x^2}\Big]_{-\frac{1}{2}}^{0}$$

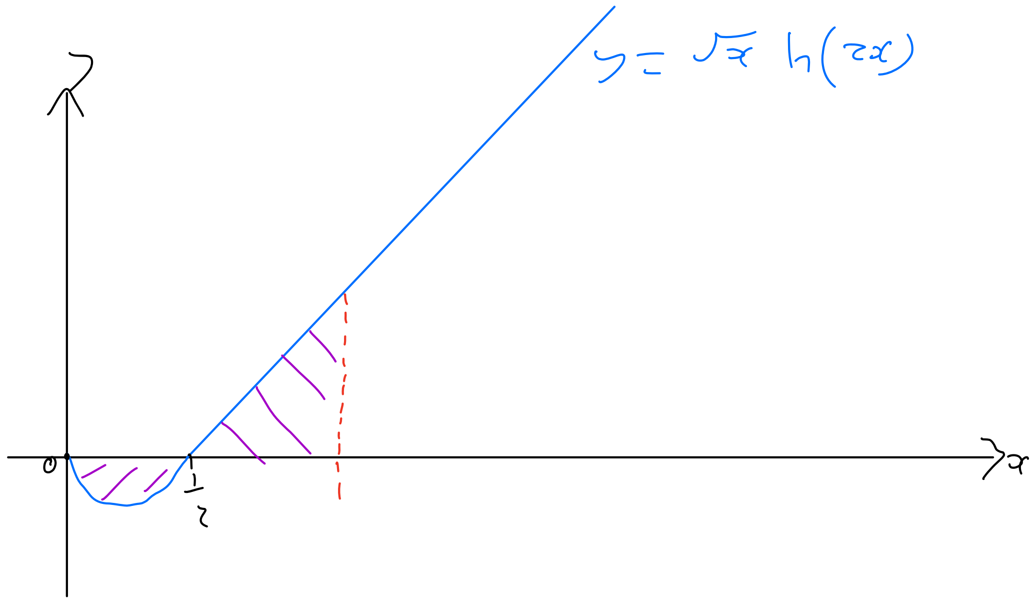
$$= 2 - \left(-2\arcsin(-1) + 2\sqrt{0}\right)$$

$$= 2 + 2\arcsin(-1) = 2 - \pi.$$

6. (a) Sketch the graph of the curve $y = \sqrt{x}\ln(2x)$, labelling all intersections with the axes.

(b) Find the area bounded by the curve $y = \sqrt{x}\ln(2x)$ and the line $x = 1$.

a)




b) Area $= \int_{\frac{1}{2}}^{1} \sqrt{x}\ln 2x \, dx - \int_{0}^{\frac{1}{2}} \sqrt{x}\ln 2x \, dx$

Consider $\int_{\frac{1}{2}}^{1} \sqrt{x}\ln 2x \, dx$

let $u = \ln 2x$ , let $\frac{dv}{dx} = x^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{x}$ , $v = \frac{2}{3}x^{\frac{3}{2}}$

$\Rightarrow \int_{\frac{1}{2}}^{1} \sqrt{x}\ln 2x \, dx = \left[\frac{2}{3}x^{\frac{3}{2}}\ln 2x\right]_{\frac{1}{2}}^{1} - \frac{2}{3}\int x^{\frac{1}{2}} \, dx$

$$= \left[ \frac{2}{3} x^{\frac{3}{2}} \ln 2x - \frac{4}{9} x^{\frac{3}{2}} \right]_{\frac{1}{2}}^{1}$$

$$= \left( \frac{2}{3} \ln 2 - \frac{4}{9} \right) - \left( \frac{2}{3} \frac{\sqrt{2}}{4} \ln 1 - \frac{4}{9} \cdot \frac{\sqrt{2}}{4} \right)$$

$$= \frac{2}{3} \ln 2 - \frac{4}{9} + \frac{\sqrt{2}}{9} = \frac{2}{3} \ln 2 + \frac{\sqrt{2} - 4}{9}.$$

Consider $\int_0^{\frac{1}{2}} \sqrt{x} \ln 2x \, dx$

$$= \left[ \frac{2}{3} x^{\frac{3}{2}} \ln 2x - \frac{4}{9} x^{\frac{3}{2}} \right]_0^{\frac{1}{2}}$$

$$= \left( \frac{2}{3} \frac{\sqrt{2}}{4} \ln 1 - \frac{4}{9} \cdot \frac{\sqrt{2}}{4} \right) - ( 0 )$$

$$= - \frac{\sqrt{2}}{9}.$$

Hence Area $= \frac{2}{3} \ln 2 - \frac{4}{9} + \frac{\sqrt{2}}{9} + \frac{\sqrt{2}}{9}$

$= \frac{2}{3} \ln 2 - \frac{4 - 2\sqrt{2}}{9}$ units$^2$.